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ON WEAK INJECTIVITY AND WEAK PROJECTIVITY

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Abstract

Given a right R-module M, a module $Q \in \sigma[M]$ is said to be weakly injective in $\sigma[M]$ if for every finitely generated submodule N of the M-injective hull $\widehat{Q}$, N is contained in a submodule Y of $\widehat{Q}$ such that $Y \simeq Q$. Weakly projective modules in $\sigma[M]$ are defined dually.

1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unitary. We denote the category of all right R-modules by Mod-R and for any $M \in$ Mod-R, $\sigma[M]$ stands for the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [12]). Given a module X_R the injective hull of X in Mod-R (resp., in $\sigma[M]$) is denoted by

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$E(X)$ (resp., $\widehat{X}$). The purpose of this paper is to further the study of the concepts of weak injectivity (projectivity) in $\sigma[M]$ studied in [3], and [13].

Given two modules Q and $N \in \sigma[M]$, we call Q weakly N - injective in $\sigma[M]$ if for every homomorphism $\varphi : N \to \widehat{Q}$, there exists a homomorphism $\widehat{\varphi} : N \to Q$ and a monomorphism $\sigma : Q \to \widehat{Q}$ such that $\varphi = \sigma\widehat{\varphi}$. Equivalently, there exists a submodule X of $\widehat{Q}$ such that $\varphi(N) \subset X \simeq Q$. A module $Q \in \sigma[M]$ is called *weakly injective* in $\sigma[M]$ if for every finitely generated submodule N of the M-injective hull $\widehat{Q}$, N is contained in a submodule Y of $\widehat{Q}$ such that $Y \simeq Q$. Equivalently, if Q is weakly N-injective for all finitely generated modules N in $\sigma[M]$.

A module X is *N-tight* in $\sigma[M]$ if every quotient of N which is embeddable in the M-injective hull of X is embeddable in X. A module X is *tight* in $\sigma[M]$ if it is tight in $\sigma[M]$ relative to all finitely generated submodules of its M-injective hull.

Given two modules Q and $N \in \sigma[M]$, we call Q *weakly N-projective* in $\sigma[M]$ if for every homomorphism $\varphi : P(Q) \to N$, where $P(Q)$ is the $\sigma[M]$-projective cover, there exists a homomorphism $\widehat{\varphi} : Q \to N$ and an epimorphism $\sigma : P(Q) \to Q$ such that $\varphi = \widehat{\varphi}\sigma$. Equivalently, if for every homomorphism $\varphi : P(Q) \to N$, there exists a submodule X of $ker(\varphi)$ such that $P(Q)/X \simeq Q$. A module $Q \in \sigma[M]$ is called *weakly projective* in $\sigma[M]$ if it is weakly N- projective for all finitely M-generated modules N in $\sigma[M]$. Given two modules Q and $N \in \sigma[M]$, we call Q *N-cotight* in $\sigma[M]$ if for every epimorphism $\varphi : P(Q) \to N$, where $P(Q)$ is the $\sigma[M]$-projective cover, there exists an epimorphism $\widehat{\varphi} : Q \to N$.

2. Weak- Projectivity (Cotightness) in $\sigma[M]$.

In this section we study some of the basic results on weak projectivity (cotightness) in $\sigma[M]$.

Theorem 2.1. Let $N, Q \in \sigma[M]$. If Q has a projective cover P in $\sigma[M]$ via an epimorphism $\pi : P \to N$. Then Q is N-projective in $\sigma[M]$ if and only if for every homomorphism $\varphi : P \to N$, there exists $\widehat{\varphi} : Q \to N$ such that $\widehat{\varphi}\pi = \phi$. Equivalently, $\phi(ker\pi) = 0$.

Proof Only if direction. Let $\varphi : P \to N$ be a homomorphism. We shall first show that $\varphi(Ker\pi) = 0$. Let $T = \varphi(Ker\pi)$ and let $\pi_T : N \to N/T$ be the natural projection. Then φ induces $\overset{\wedge}{\varphi}\,(q) = \pi_T\varphi(p)$, where $q = \pi(p)$. It follows that $\overset{\wedge}{\varphi}\,\pi = \pi_T\varphi$. Since Q is $N-$projective in $\sigma[M]$, there exists a map $\beta : Q \to N$ such that $\overset{\wedge}{\varphi} = \pi_T\varphi$. Clearly, $(\varphi - \beta\pi)P \subseteq T$. We claim that $\varphi = \beta\pi$

Let $X = \{p \in P|\varphi(p) = \beta\pi(p)\}$. We shall show that $X = P$. Let $x \in P$. Since $(\varphi - \beta\pi)(x) \in T = \varphi(Ker\pi)$, there exists $k \in Ker\pi$ such that $(\varphi-\beta\pi)(x) = \varphi(k)$. Therefore , $(\varphi-\beta\pi)P = 0$. In particular, $(\varphi-\beta\pi)Ker\pi = 0$, yielding $\varphi(\ker\pi) = 0$. Equivalently, there exists $\varphi' : Q \to N$ such that $\varphi'\pi = \varphi$.

Conversely, let $\Phi : Q \to N/K$ be a homomorphism and $\Phi' : P \to N$ such that $\Phi\pi = \pi_K\Phi'$. By our hypothesis there exists $\Phi'' : Q \to N$ such that $\Phi''\pi = \Phi'$. It follows easily that $\pi_K\Phi'' = \Phi$, proving that Q is $N-$projective.

The next theorem is a very useful characterization of weak projectivity.

Theorem 2.2. Let $N, Q \in \sigma[M]$. If Q has a projective cover P in $\sigma[M]$ via an epimorphism $\pi : P \to N$. Then Q is weakly N-projective in $\sigma[M]$ if and only if for every homomorphism $\varphi : P \to N$ there exists $X \subset ker\varphi$ such that $P/X \simeq Q$.

Proof Let $\varphi : P \to N$ be a homomorphism. Assume that Q is weakly $N-$ projective in $\sigma[M]$ and let $\widehat{\varphi} : Q \to N$ be the homomorphism and $\sigma : P \to N$ the epimorphism as in the definition of weak $N-$projectivity. Since $\varphi = \widehat{\varphi}\sigma$, $\ker\sigma \subseteq \ker\varphi$. Thus the implication is proven by taking $X = \ker\sigma$. Conversely, if $X \subseteq P$ satisfies the conditions in the statement of the theorem, then the isomorphism $P/X \cong Q$, composed with the natural projection $\pi_X : P \to P/X$ is an epimorphism σ satisfies that $\ker\sigma = X \subseteq \ker\varphi$. It follows that the mapping $\widehat{\varphi} : Q \to N$ given by $\widehat{\varphi}(q) = \varphi(p)$, whenever $\sigma(p) = q$ is well-defined and satisfies $\varphi = \widehat{\varphi}\sigma$, proving that Q is weakly $N-$projective.

For cotightness, following similar proof as in Theorem 2.2 one gets the following characterization.

Theorem 2.3. Let $N, Q \in \sigma[M]$. If Q has a projective cover P in $\sigma[M]$ via an epimorphism $\pi : P \to N$. Then Q is N-cotight in $\sigma[M]$ if and only if for every homomorphism $\varphi : P \to N$ there exists $X \subset ker\varphi$ and $K \subset Q$

such that $P/X \simeq Q/K$.

The class of weak projectivity in $\sigma[M]$ is closed under submodules and quotient modules.

Proposition 2.4. For modules $N, L \in \sigma[M]$, the following conditions are equivalent:

(a) L is weakly N-projective in $\sigma[M]$;

(b) L is weakly K-projective in $\sigma[M]$ for every submodule K of N;

(c) L is weakly N/K-projective in $\sigma[M]$ for every submodule K of N;

(d) for every submodule K of N, and for every epimorphism $\varphi : P(L) \to K$, where $P(L)$ is the $\sigma[M]$-projective cover, there exists an epimorphism $\widehat{\varphi} : K \to L$ and an epimorphism $\sigma : P(L) \to L$ such that $\varphi = \widehat{\varphi}\sigma$.

Proof$(a) \Rightarrow (b)$. Assume L is weakly N-projective and let $\varphi : P(L) \to K$ be a homomorphism. By weak projectivity of L, $f = i_K\varphi$ factors through L by an epimorphism $\sigma : P(L) \to L$ and a homomorphism $\widehat{f} : L \to N$. Since σ is onto, the range of $\widehat{f}$ equals the range of f and so is contained in K. Define $\widehat{\varphi} : Q \to K$ by $\widehat{\varphi}(q) = \widehat{f}(q)$. Then it follows that $\varphi = \widehat{\varphi}\sigma$.

$(b) \Rightarrow (c)$. Let K be a submodule of N and let $\varphi : P(L) \to N/K$ be a homomorphism. By the projectivity of $P(L)$,there exists a homomorphism $\widehat{\varphi} : P(L) \to N$ such that $\varphi = \pi_K\widehat{\varphi}$. Since L is weakly $N-$projective, there exists an epimorphism $\sigma : P(L) \to L$ and a homomorphism $\widehat{\sigma} : L \to N$ such that $\widehat{\varphi} = \widehat{\sigma}\sigma$. It follows that $\varphi = \pi_K\widehat{\sigma}\sigma$, proving that L is weakly $N/K - projective.$

$(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$ are straightforward.

Finite direct sums of weakly projectives (cotights) in $\sigma[M]$ and superfluous covers of weakly projective modules are also weakly projective in $\sigma[M]$.

Proposition 2.5. For modules N, L and $K \in \sigma[M]$, we have the following:

(a) if L and K are weakly N-projective (cotight) in $\sigma[M]$, then $L \oplus K$ is weakly N-projective (cotight) in $\sigma[M]$,

(b) if L is weakly N-projective in $\sigma[M]$ and K is a superfluous cover of L then K is weakly N-projective in $\sigma[M]$,

(c) if a module X in $\sigma[M]$ is weakly projective relative to its projective cover in $\sigma[M]$, then X is projective in $\sigma[M]$. Consequently, a finitely generated weakly projective module in $\sigma[M]$ is indeed projective in $\sigma[M]$.

Proof Straightforward from the definition.

Proposition 2.6. Let $\{X_i\}_I$ be a class of weakly N-projectives (cotight) in $\sigma[M]$ and $\bigoplus_I X_i$ has a projective cover in $\sigma[M]$. Then $\bigoplus_I X_i$ is weakly N-projective (cotight) in $\sigma[M]$.

ProofThe proof follows directly from the fact that in this case $P(\bigoplus_I X_i) = \bigoplus_I P(X_i)$.

The next proposition shows the difference between weak-projectivity and cotightness in $\sigma[M]$ and the proof is the same as in [6].

Proposition 2.7. Given modules $N, Q \in \sigma[M]$, and assume Q is supplemented and has a projective cover $\pi : P \to Q$ in $\sigma[M]$. Then Q is weakly N-projective in $\sigma[M]$ if and only if for every submodule K of N and for every epimorphism $\varphi : P \to K$, there exists an epimorphism $\widehat{\varphi} : Q \to K$ such that for every supplement L' of $ker\widehat{\varphi}$ in Q, there exists a submodule L of P such that $P/L \simeq Q/L'$ and $L + ker\varphi = P$.

Corollary 2.8. Given modules $N, Q \in \sigma[M]$. If Q is hollow then Q is N-cotight in$\sigma[M]$ iff Q is weakly N-projective in $\sigma[M]$.

Proposition 2.9. Given modules $N, Q \in \sigma[M]$. If Q is self-projective and cotight (weakly N-projective) in $\sigma[M]$, then Q is indeed N-projective in $\sigma[M]$.

ProofLet $\varphi : P \to K$. Since Q is cotight in $\sigma[M]$ there exists an epimorphism $\widehat{\varphi} : Q \to Im(\varphi)$ and by the projectivity of P, there exists a homomorphism $f : P \to Q$ such that $\varphi = \widehat{\varphi} f$. By self-projectivity of Q and Theorem 3.1, there exists a homomorphism $\widehat{f} : Q \to Q$ such that $f = \widehat{f}\pi$. It follows that $\varphi = \widehat{\varphi}\widehat{f}\pi$, proving that Q is $N-$projective.

A finitely generated direct summand S of the projective cover of a weakly projective (cotight) module X in $\sigma[M]$ yields a direct summand (isomorphic to S) of X.

Proposition 2.10. Let Q be a (cotight) weakly projective module in $\sigma[M]$ whose projective cover in $\sigma[M]$ has a finitely generated direct summand

S. Then Q has a direct summand isomorphic to S.

ProofSince S is finitely generated, Q is $S-$cotight. Thus the projection map $\pi_S : P(Q) \to S$ yields an epimorphism $\pi'_S : Q \to S$. Since S is projective we get $Q \cong S \oplus \ker \pi'_S$, proving our claim.

Proposition 2.11. Let M_R be locally noetherian, and let Q, N be finitely generated in $\sigma[M]$. If Q is N-cotight in $\sigma[M]$ and N is Q-cotight in $\sigma[M]$ and $Q/J(Q) \simeq N/J(N)$ then $Q \simeq N$.

ProofLet $\sigma : P(Q) \to N$ be the epimorphism induced by the isomorphism between $Q/J(Q)$ and $N/J(N)$. Since Q is N-cotight in $\sigma[M]$, N is a homomorphic image of M. Similarly, M is a homomorphic image of N. Since M and N are finitely generated over artinian ring, $M \simeq N$.

3. Weak- Injectivity (tightness) in $\sigma[M]$.

In this section we dualize most of the basic results on weak projectivity in $\sigma[M]$ given in the previous section and the proof is dualizable in most of these cases.

Proposition 3.1. Let $Q, N \in \sigma[M]$. Then Q is weakly N-injective in $\sigma[M]$ if and only if for every homomorphism $\varphi : N \to \widehat{Q}$, there exists a submodule X of $\widehat{Q}$ such that $\varphi(N) \subset X \simeq Q$.

The class of weak injectivity in $\sigma[M]$ is closed under submodules and quotient modules as it is shown in the next proposition.

Proposition 3.2. For modules N, $L \in \sigma[M]$, the following conditions are equivalent:

(a) L is weakly N-injective in $\sigma[M]$;

(b) L is weakly K-injective in $\sigma[M]$ for every submodule K of N;

(c) L is weakly N/K-injective in $\sigma[M]$ for every submodule K of N;

(d) for every submodule K of N, and for every monomorphism $\varphi : N/K \to \widehat{L}$, there exists a monomorphism $\widehat{\varphi} : N/K \to L$ and a monomorphism $\sigma : N/K \to \widehat{L}$ such that $\varphi = \sigma\widehat{\varphi}$.

Proposition 3.3. For modules N, L and $K \in \sigma[M]$, we have the following:

(a) if L and K are weakly N-injective (tight) in $\sigma[M]$ then $L \oplus K$ is weakly N-injective (tight) in $\sigma[M]$,

(b) if L is weakly N-injective in $\sigma[M]$ and L is an essential submodule of K then K is weakly N-injective in $\sigma[M]$.

Proposition 3.4. Given modules $N, Q \in \sigma[M]$, Q is weakly N-injective in $\sigma[M]$ if and only if for every submodule K of N and for every monomorphism φ: $N/K \to \widehat{Q}$, there exists a monomorphism $\widehat{\varphi} : N/K \to Q$, and for every complement L of $\widehat{\varphi}(N/K)$ in Q, there exists $L' \subset \widehat{Q}$ such that $L' \cap \varphi(N/K) = 0$ and $L' \simeq L$.

Corollary 3.5. Given modules $N, Q \in \sigma[M]$. If Q is uniform then Q is N-tight in $\sigma[M]$ iff Q is weakly N-injective in $\sigma[M]$.

Proposition 3.6. Given modules $N, Q \in \sigma[M]$. If Q is self-injective and N-tight (weakly $N-$injective) in $\sigma[M]$, then Q is indeed N-injective in $\sigma[M]$.

Proposition 3.7. Let M be a locally artinian module, and let N, Q be finitely generated modules in $\sigma[M]$. If Q is N-tight in $\sigma[M]$ and N is Q-tight in $\sigma[M]$ and $Soc(Q) \simeq Soc(N)$ then $Q \simeq N$.

ProofLet $\sigma : N \to E(M)$ be the monomorphism induced by the isomorphism between $Soc(M)$ and $Soc(N)$. Since M is N-tight in $\sigma[M]$, N is embeddable in M. Similarly, M is embeddable in N. Since M and N are finitely generated over artinian ring, $M \simeq N$.

4. A Characterization of Semisimple Modules.

In this section we characterize (weakly) semisimple modules by weak projectivity and weak injectivity in $\sigma[M]$.

Lemma 4.1. Let M be projective and perfect in $\sigma[M]$. Then there exists a module $K \in \sigma[M]$ such that $K \oplus X$ is a weakly projective module in $\sigma[M]$, for every module $X \in \sigma[M]$.

Lemma 4.2. Every semisimple module in $\sigma[M]$ is a direct summand of a weakly injective module in $\sigma[M]$.

Lemma 4.3. Every module in $\sigma[M]$ is a direct summand of a tight module in $\sigma[M]$.

Lemma 4.4. Let M be a locally *q.f.d.* module. Then every module $K \in \sigma[M]$ is a direct summand of a weakly injective module $Q = K \oplus (\widehat{K})^{(\alpha)}$ in $\sigma[M]$, where α is an infinite cardinal number.

The proof of the next theorem follows easily from the above results.

Theorem 4.5. For a module M_R. The following are equivalent:

(a) M is semisimple;

(b) M is projective and perfect and every weakly projective module in $\sigma[M]$ is (quasi-) discrete;

(c) M is projective and perfect and every discrete module is weakly projective in $\sigma[M]$;

(d) M is projective and perfect and every weakly projective module in $\sigma[M]$ is (quasi-) continuous;

(e) M is locally *q.f.d.* and every weakly injective module in $\sigma[M]$ is (quasi-) discrete;

(f) M is locally *q.f.d.* and every weakly injective module in $\sigma[M]$ is (quasi-) continuous;

(g) every continuous module is weakly projective in $\sigma[M]$;

(h) every (direct summand of a) weakly injective module in $\sigma[M]$ is (injective) projective in $\sigma[M]$;

(i) M is projective and perfect and every weakly projective module in $\sigma[M]$ is injective (projective) in $\sigma[M]$;

(j) M is projective and perfect in $\sigma[M]$ and every direct summand of a weakly projective module in $\sigma[M]$ is weakly projective in $\sigma[M]$;

(k) M is projective and perfect in $\sigma[M]$ and every (direct summand of a) weakly projective module in $\sigma[M]$ is quasi-projective in $\sigma[M]$;

(l) every direct summand of a weakly injective module in $\sigma[M]$ is quasi-injective in $\sigma[M]$;

(m) M is projective and perfect in $\sigma[M]$ and every direct summand of a weakly projective module in $\sigma[M]$ is injective in $\sigma[M]$.

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